W8/2 Integral Extensions

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6.1 Cohen-Seidenberg Theorems

Let $A \subseteq B$ be an integral ring extension. Goal: Relate $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$.

Trivial: If $Q \in \mathrm{Spec}(B)$, then $P := A \cap Q \in \mathrm{Spec}(A)$

Then Q lies over P, P is under Q.

Prop 6.7 Let $A \subseteq B$ integral.

(1) Suppose A, B are domains. Then: A field $\Leftrightarrow$ B field

(2) Let $Q \in \mathrm{Spec}(B)$: $\quad Q \in \mathrm{Max}(B) \Leftrightarrow P := Q \cap A \in \mathrm{Max}(A)$

Proof: (1) „$\Rightarrow$" Let $0 \neq b \in B$. Let $0 = b^n + a_{n-1} b^{n-1} + \dots + a_0$ $\quad (a_i \in A)$ be an integral eqn. for b of minimal degree. Then $a_0 \neq 0$ (otherwise, can cancel some b's since B is a domain).

$$\Rightarrow a_0 \in A^\times \Rightarrow b\left(-\frac{1}{a_0} b^{n-1} - \frac{a_{n-1}}{a_0} b^{n-2} - \dots - \frac{a_1}{a_0}\right) = 1 \Rightarrow b \in B^\times.$$

„$\Leftarrow$" Let $0 \neq a \in A \Rightarrow a^{-1} \in B \Rightarrow a^{-1}$ is integral $/A$

$$\Rightarrow a^{-n} + c_{n-1} a^{-n+1} + \dots + c_0 = 0 \quad (c_i \in A) \qquad \Big| \cdot a^{n-1}$$

$$\Rightarrow a^{-1} = -c_{n-1} - c_{n-2} a - \dots - c_0 a^{n-1} \in A \Rightarrow a \in A^\times.$$

(2) $A/P \hookrightarrow B/Q$ is an integral extension (P6.5(1)) of domains. Apply (1). $\square$

Thm 6.8 Let $A \subseteq B$ be integral.

(1) (Lying Over) If $P \in \mathrm{Spec}(A)$, there exists $Q \in \mathrm{Spec}(B)$ s.t. $Q \cap A = P$

(2) (Incomparability) If $Q, Q' \in \mathrm{Spec}(B)$ lie over over $P := Q \cap A = Q' \cap A$, and $Q \subseteq Q'$, then $Q = Q'$.

(3) (Going Up) Let $P_1 \subseteq P_2 \subseteq \dots \subseteq P_n$ in $\mathrm{Spec}(A)$ $(n \geq 1)$. If $Q_1 \subseteq \dots \subseteq Q_m$ $(m < n)$ in $\mathrm{Spec}(B)$ is s.t. $Q_i \cap A = P_i$, there exist $Q_{m+1}, \dots, Q_n \in \mathrm{Spec}(B)$ s.t. $Q_1 \subseteq \dots \subseteq Q_n$ and $Q_i \cap A = P_i$ for all $1 \leq i \leq n$.

Proof: (1) Let $S = A \setminus P$, $j: B \to B_P$ the localization.

$$\begin{array}{ccc} B & \xrightarrow{j} & B_P \\ \cup\! | & & \cup\! | \\ A & \xrightarrow{j|_A} & A_P \end{array}$$

$B_P := S^{-1}B$ is integral $/A_P$ by P6.5(2). Let $M \in \mathrm{Max}(B_P)$

Proof: (1) Let $S = A \setminus P$, $j: B \to B_p$ the localization.

$$\begin{matrix} \cup\!\mathsf{I} & & \cup\!\mathsf{I} \\ A & \xrightarrow[j|_A]{} & A_p \end{matrix}$$

$B_p := S^{-1}B$ is integral $/A_p$ by P6.5(2). Let $M \in Max(B_p)$

$\Rightarrow M \cap A_p$ is maximal in A_p by P6.7(2) $\overset{A_p \text{ local}}{\Longrightarrow} M \cap A_p = PA_p$

$\Rightarrow j^{-1}(M) \in Spec(B)$, $\quad j^{-1}(M) \cap A = j^{-1}(M) \cap j^{-1}(A_p) = j^{-1}(M \cap A_p) = j^{-1}(PA_p) = P.$

(2) Localize: $PA_p \in Max(A_p)$, $QB_p, Q'B_p \in Spec(B)$

$\Rightarrow QB_p \subseteq Q'B_p$ and $QB_p \cap A_p = PA_p = Q'B_p \cap A_p$

$\xrightarrow[P6.7(2)]{B_p \text{ integral}/A_p}$ $QB_p, Q'B_p \in Max(B_p) \Rightarrow QB_p = Q'B_p \Rightarrow Q = j^{-1}(QB_p) = j^{-1}(Q'B_p) = Q'$.

(3) By induction w.r.t. $m=1$, $n=2$.

$\bar{A} := A/P_1$, $\bar{B} := B/Q_1 \Rightarrow \bar{B}$ integral over $\bar{A}$ (P6.7(2))

$\bar{P}_2 := P_2/P_1 \in Spec(\bar{A})$. $\quad (1) \Rightarrow \exists \bar{Q}_2 \in Spec(\bar{B}) : \bar{Q}_2 \cap \bar{A} = \bar{P}_2$

Lift $\bar{Q}_2$ to Q_2 under $B \to \bar{B}$. $\square$

Cor 6.9 Let $A \subseteq B$ be integral

(1) $\dim(A) = \dim(B)$

(2) $A^\times = B^\times \cap A$.

Proof: (1) If $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ is a chain in $Spec(A)$, GU yields a chain $Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n$ in $Spec(B)$ with $Q_i \cap A = P_i$. In particular, $Q_i \subsetneq Q_{i+1}$, so $\dim(B) \geq \dim(A)$.

Conversely, if $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_n$ is a chain in $Spec(B)$, let $P_i := Q_i \cap A$. Then $P_0 \subseteq \ldots \subseteq P_n$ is a chain in $Spec(A)$. By INC $P_i \subsetneq P_{i+1}$. So $\dim(A) \geq \dim(B)$.

(2) "$\subseteq$" ✓ "$\supseteq$": Let $a \in A \cap B^\times$. Suppose $a \notin A^\times \Rightarrow \exists M \in Max(A): a \in M$

$\overset{LO}{\Longrightarrow} \exists Q \in Max(B): a \in M \subseteq Q$ ↯ $\square$

Def: $A \subseteq B$ ring ext., $I \trianglelefteq A$.

$b \in B$ is *integral over I* if $b^n + a_{n-1}b^{n-1} + \ldots + a_0 = 0$ with $a_0, \ldots, a_{n-1} \in I$.

The *integral closure* of I in B, $cl_B(I)$, is the set of all such elements.

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<u>Prop 6.10</u> Let $A \subseteq B$ be rings, $I \trianglelefteq A$.

Then $cl_B(I) = \sqrt{I \cdot cl_B(A)}$

<u>Proof:</u> "$\subseteq$": Let $b \in B$, $b^n + a_{n-1} b^{n-1} + \ldots + a_0 = 0$ with $a_i \in I$

$\Rightarrow b^n \in I \cdot cl_B(A) \rightarrow b \in \sqrt{I \cdot cl_B(A)}$.

"$\supseteq$" Let $b \in \sqrt{I \cdot cl_B(A)} \Rightarrow b^n = x_1 b_1 + \ldots + x_m b_m$ with $x_i \in I$, $b_i \in cl_B(A)$

b_i integral $\overset{C6.2}{\Rightarrow} A' := A[b_1, \ldots, b_m]$ is a f.g. A-module.

Then $b^n A' \subseteq I A'$, and the determinant trick (cf. P6.1) produces $a_0, \ldots, a_{k-1} \in I$ s.t. $b^k + a_{k-1} b^{k-1} + \ldots + a_0 = 0$. (Details skipped).

$\square$